MEE304: DYNAMICS OF MACHI ERY (2 UNITS)

(LECTURE NOTE)



ENGR. DR. I. S. SINTALI

Department of Mechanical Engineering

1.0 INTRODUCTION

The high speed of engines and other machines is a common phenomenon now-a-days. It is, therefore, very essential that all the rotating and reciprocating parts should be completely balanced as far as possible. If these parts are not properly balanced, the dynamic forces are set up. These forces not only increase the loads on bearings and stresses in the various members, but also produce unpleasant and even dangerous vibrations. Thus, we shall discuss the balancing of unbalanced forces caused by rotating and reciprocating parts, in order to minimize pressure on the main bearings when an engine is running.

1.1 Balancing of Rotating Masses

Whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it. In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal forces of both the masses are made to be equal and opposite. The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass. The following cases are important from the subject point of view:

- i. Balancing of a single rotating mass by a single mass rotating in the same plane.
- ii. Balancing of a single rotating mass by two masses rotating in different planes.
- iii. Balancing of different masses rotating in the same plane.
- iv. Balancing of different masses rotating in different planes.

We shall now discuss these cases, in detail.

1.1.1 Balancing of a Single Rotating Mass By a Single Mass Rotating in the Same Plane

Consider a disturbing mass m_1 attached to a shaft rotating at $\omega rad/s$ as shown in Figure 1. Let r_1 be the radius of rotation of the mass m_1 (*i.e.* distance between the axis of rotation of the shaft and the center of gravity of the mass m_1). We know that the centrifugal force exerted by the mass m_1 on the shaft,

$$F_{C_1} = m_1 \omega^2 r_1 \qquad \dots (1)$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft. In order to counteract the effect of this force, a balancing mass (m_2) may be attached in the same plane of rotation as that of disturbing mass (m_1) such that the centrifugal forces due to the two masses are equal and opposite.



Figure 1: Balancing of a single rotating mass by a single mass rotating in the same plane

Let $r_2 = \text{Radius of rotation of the balancing mass } m_2$ (*i.e.* distance between the axis of rotation of the shaft and the center of gravity of mass m_2).

Therefore centrifugal force due to mass m_2 ,

$$F_{C_2} = m_2 \,\omega^2 r_2 \qquad \dots (2)$$

Equating equations (1) and (2),

$$m_1 \omega^2 r_1 = m_2 \omega^2 r_2$$
 or $m_1 r_1 = m_2 r_2$

Notes:

- i. The product $m_2 r_2$ may be split up in any convenient way. But the radius of rotation of the balancing mass (m_2) is generally made large in order to reduce the balancing mass m_2 .
- ii. The centrifugal forces are proportional to the product of the mass and radius of rotation of respective masses, because ω^2 is same for each mass.

1.1.2 Balancing of a Single Rotating Mass by Two Masses Rotating in Different Planes

We have discussed in the previous article that by introducing a single balancing mass in the same plane of rotation as that of disturbing mass, the centrifugal forces are balanced. In other words, the two forces are equal in magnitude and opposite in direction. But this type of arrangement for balancing gives rise to a couple which tends to rock the shaft in its bearings. Therefore in order to put the system in complete balance, two balancing masses are placed in two different planes, parallel to the plane of rotation of the disturbing mass, in such a way that they satisfy the following two conditions of equilibrium.

- i. The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the center of the masses of the system must lie on the axis of rotation. This is the condition for *static balancing*.
- ii. The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

The conditions (i) and (ii) gives *dynamic balancing*; the following two possibilities may arise while attaching the two balancing masses:

- i. The plane of the disturbing mass may be in between the planes of the two balancing masses, and
- ii. The plane of the disturbing mass may lie on the left or right of the two planes containing the balancing masses.

We shall now discuss both the above cases one by one.

1.1.2.1 When the plane of the disturbing mass lies in between the planes of the two balancing masses

Consider a disturbing mass *m* lying in a plane *A* to be balanced by two rotating masses m_1 and m_2 lying in two different planes *L* and *M* as shown in the Figure 2. Let *r*, r_1 and r_2 be the radii of rotation of the masses in planes *A*, *L* and *M* respectively.

Let l_1 = Distance between the planes A and L,

 l_2 = Distance between the planes *A* and *M*, and

l =Distance between the planes L and M.



Figure 2: Balancing of a single rotating mass by two rotating masses in different planes when the plane of single rotating mass lies in between the planes of two balancing masses

We know that the centrifugal force exerted by the mass *m* in the plane *A*,

$$F_c = m\omega^2 r$$

Similarly, the centrifugal force exerted by the mass m_1 in the plane L,

$$F_{C_1} = m_1 \omega^2 r_1$$

and the centrifugal force exerted by the mass m_2 in the plane M,

$$F_{C_2} = m_2 \omega^2 r_2$$

Since the net force acting on the shaft must be equal to zero, therefore the centrifugal force on the disturbing mass must be equal to the sum of the centrifugal forces on the balancing masses, therefore

$$F_{c} = F_{C_{1}} + F_{C_{2}} \qquad m\omega^{2} r = m_{1}\omega^{2} r_{1} + m_{2}\omega^{2} r_{2}$$

$$m r = m_{1} r_{1} + m_{2} r_{2} \qquad \dots (i)$$

Now in order to find the magnitude of balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{c_1} \times l = F_c \times l_2$$
 or $m_1 \cdot \omega^2 \cdot r_1 \times l = m \cdot \omega^2 \cdot r \times l_2$

:.

$$m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2$$
 or $m_1 \cdot r_1 = m \cdot r \cdot \frac{l_2}{l}$... (ii)

Similarly, in order to find the balancing force in plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{c_2} \times l = F_c \times l_1 \quad or \quad m_2 . \omega^2 . r_2 \times l = m . \omega^2 . r \times l_1$$
$$m_2 . r_2 . l = m . r . l_1 \quad or \quad m_2 . r_2 = m . r . \frac{l_1}{l} \qquad \dots \text{(iii)}$$

It may be noted that equation (*i*) represents the condition for static balance, but in order to achieve dynamic balance, equations (*ii*) or (*iii*) must also be satisfied.

1.1.2.2. When the plane of the disturbing mass lies on one end of the planes of the balancing masses



Figure 3: Balancing of a single rotating mass by two rotating masses in different planes, when the plane of single rotating mass lies at one end of the planes of balancing masses

In this case, the mass m lies in the plane A and the balancing masses lay in the planes L and M, as shown in the Figure 3. As discussed above, the following conditions must be satisfied in order to balance the system, *i.e.*

$$F_{c} + F_{c_{2}} = F_{c_{1}} \qquad m\omega^{2} r + m_{2}\omega^{2} r_{2} = m_{1}\omega^{2} r_{1}$$

$$m r + m_{2} r_{2} = m_{1} r_{1} \qquad \dots \text{(iv)}$$

Now, to find the balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{C_1} \times l = F_c \times l_2 \quad or \quad m_1 \omega^2 r_1 \times l = m \omega^2 r \times l_2$$
$$m_1 r_1 \cdot l = m r \cdot l_2 \quad or \quad m_1 r_1 = m r \times \frac{l_2}{l} \qquad \dots (v)$$

:.

... [Same as equation (ii)]

Similarly, to find the balancing force in the plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{C_2} \times l = F_c \times l_1 \quad or \quad m_2 \omega^2 r_2 \times l = m \omega^2 r \times l_1$$

$$\therefore \qquad m_2 r_2 \cdot l = m r \cdot l_1 \quad or \quad m_2 r_2 = m r \times \frac{l_1}{l} \qquad \dots \text{(vi)}$$

... [Same as equation (iii)]

1.1.3 Balancing of Several Masses Rotating in the Same Plane

Consider any number of masses (say four) of magnitude m_1 , m_2 , m_3 and m_4 at distances of r_1 , r_2 , r_3 and r_4 from the axis of the rotating shaft. Let θ_1 , θ_2 , θ_3 and θ_4 be the angles of these masses with the horizontal line *OX*, as shown in Figure 4 (a) below. Let these masses rotate about an axis through *O* and perpendicular to the plane of paper, with a constant angular velocity of $\omega rad/sec$. The magnitude and position of the balancing mass may be found out analytically or graphically as discussed in Figure 4 (b) below:



Figure 4: Balancing of several masses rotating in the same plane

1.1.3.1. Analytical method

The magnitude and direction of the balancing mass may be obtained, analytically, as discussed below:

i. First of all, find out the centrifugal force* (or the product of the mass and its radius of rotation) exerted by each mass on the rotating shaft.

ii. Resolve the centrifugal forces horizontally and vertically and find their sums, *i.e.* Σ H and Σ V. We know that

Sum of horizontal components of the centrifugal forces,

 $\Sigma \mathbf{H} = \mathbf{m}_1 r_1 cos \theta_1 + \mathbf{m}_2 r_2 cos \theta_2 + \cdots \dots$

and sum of vertical components of the centrifugal forces,

 $\Sigma V = m_1 r_1 sin\theta_1 + m_2 r_2 sin\theta_2 + \cdots \dots$

iii. Magnitude of the resultant centrifugal force,

 $F_c = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$

- iv. If θ is the angle, which the resultant force makes with the horizontal, then $tan\theta = \Sigma V / \Sigma H$.
- v. The balancing force is then equal to the resultant force, but in *opposite direction*.
- vi. Now find out the magnitude of the balancing mass, such that

 $F_c = m.r$

Where m = Balancing mass and

r = its radius of rotation

1.1.3.2. Graphical method

The magnitude and position of the balancing mass may also be obtained graphically as discussed below:

- i. First of all, draw the space diagram with the positions of the several masses, as shown in Figure 4 (*a*).
- ii. Find out the centrifugal force (or product of the mass and radius of rotation) exerted by each mass on the rotating shaft.
- iii. Now draw the vector diagram with the obtained centrifugal forces (or the product of the masses and their radii of rotation), such that *ab* represents the centrifugal force exerted by the mass m_1 (or $m_1.r_1$) in magnitude and direction to some suitable scale. Similarly, draw *bc*, *cd* and *de* to represent centrifugal forces of other masses m_2, m_3 and m_4 (or $m_2.r_2, m_3.r_3$ and $m_4.r_4$).
- iv. Now, as per polygon law of forces, the closing side ae represents the resultant force in magnitude and direction, as shown in Figure 4 (*b*).
- v. The balancing force is, then, equal to the resultant force, but in opposite direction

- vi. Now find out the magnitude of the balancing mass (m) at a given radius of rotation (r), such that $m. \omega^2 \cdot r =$ Resultant centrifugal force, or $m.r = Resultant of m_1.r_1, m_2.r_2, m_3.r_3 and m_4.r_4$
- **Example 1:** Four masses m_1 , m_2 , m_3 and m_4 are 200 kg, 300 kg, 240 kg and 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m, and 0.3 m respectively. The angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m.

Solution: Given: $m_1 = 200 \ kg$; $m_2 = 300 \ kg$; $m_3 = 240 \ kg$; $m_4 = 260 \ kg$; $r_1 = 0.2 \ m$;

$$\begin{aligned} r_2 &= 0.15 \ m \ ; \ r_3 \ = \ 0.25 \ m \ ; \ r_4 \ = \ 0.3 \ m ; \ \theta_1 = 0^\circ \ ; \ \theta_2 = 45^\circ \ ; \ \theta_3 = 45^\circ + 75^\circ = 120^\circ ; \\ \theta_4 &= 45^\circ + 75^\circ + 135^\circ = 255^\circ \ ; \ r \ = \ 0.2 \ m \end{aligned}$$

Let m = Balancing mass, and

 θ = The angle which the balancing mass makes with m_1 .

Since the magnitude of centrifugal forces are proportional to the product of each mass and its radius, therefore,

$$m_1 r_1 = 200 \times 0.2 = 40 \ kg - m$$

 $m_2 r_2 = 300 \times 0.15 = 45 \ kg - m$
 $m_3 r_3 = 240 \times 0.25 = 60 \ kg - m$
 $m_4 r_4 = 260 \times 0.3 = 78 \ kg - m$

The problem may, now, be solved either analytically or graphically. But we shall solve the problem by both the methods one by one.

i. Analytical method

The space diagram is as shown in Figure 5 below.

Resolving $m_1 \cdot r_1, m_2 \cdot r_2, m_3 \cdot r_3$ and $m_4 \cdot r_4$ horizontally,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + m_3 \cdot r_3 \cos\theta_3 + m_4 \cdot r_4 \cos\theta_4$$

= 40\cos0^o + 45\cos45^o + 60\cos120^o + 78\cos255^o
= 40 + 31.8 - 30 - 20.2 = 21.6kg - m



Figure 5: Space diagram

Now resolving vertically,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + m_3 \cdot r_3 \sin\theta_3 + m_4 \cdot r_4 \sin\theta_4$$

= 40sin0° + 45sin45° + 60sin120° + 78ins255°
= 0 + 31.8 + 52 - 75.3 = 8.5 kg - m

: Resultant,
$$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(21.6)^2 + (8.5)^2} = 23.2 \ kg - m$$

We know that

$$m.r = R = 23.2 \text{ or } m = \frac{23.2}{r} = \frac{23.2}{0.2} = 116kg$$

and $tan\theta' = \frac{\Sigma V}{\Sigma H} = \frac{8.5}{21.6} = 0.3935$ or $\theta' = 21.48^{\circ}$

Since θ' is the angle of the resultant *R* from the horizontal mass of 200 kg, therefore the angle of the balancing mass from the horizontal mass of 200 kg,

 $\theta = 180^{\circ} + 21.48^{\circ} = 201.48^{\circ}$

ii. Graphical method

The magnitude and the position of the balancing mass may also be found graphically as discussed below:

- a) First of all, draw the space diagram showing the positions of all the given masses as shown in Figure 6 (a) below.
- b) Since the centrifugal force of each mass is proportional to the product of the mass and radius, therefore

 $m_1 r_1 = 200 \times 0.2 = 40 \ kg - m$

$$m_2 r_2 = 300 \times 0.15 = 45 \text{kg} - \text{m}$$

 $m_3 r_3 = 240 \times 0.25 = 60 \text{ kg} - \text{m}$ $m_4 r_4 = 260 \times 0.3 = 78 \text{ kg} - \text{m}$

c) Now draw the vector diagram with the above values, to some suitable scale, as shown in Figure 6 (*b*). The closing side of the polygon *ae* represents the resultant force. By measurement, we find that ae = 23 kg-m.



Figure 6: Space diagram showing the positions of all the given masses

d. The balancing force is equal to the resultant force, but *opposite* in direction as shown in Figure 6 (a). Since the balancing force is proportional to m. r, therefore m × 0.2 = vector ea = 23 kg - m or m = 23/0.2 = 115 kg Ans. By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal mass of 200 kg, θ = 201° Ans.

1.1.4. Balancing of Several Masses Rotating in Different Planes

When several masses revolve in different planes, they may be transferred to a *reference plane* (briefly written as *R.P.*), which may be defined as the plane passing through a point on the axis of rotation and perpendicular to it. The effect of transferring a revolving mass (in one plane) to a reference plane is to cause a force of magnitude equal to the centrifugal force of the revolving mass to act in the reference plane, together with a couple of magnitude equal to the product of the force and the distance between the plane of rotation and the reference plane. In order to have a complete balance of the several revolving masses in different planes, the following two conditions must be satisfied:

- The forces in the reference plane must balance, *i.e.* the resultant force must be i. zero.
- ii. The couples about the reference plane must balance, *i.e.* the resultant couple must be zero.

Let us now consider four masses m_1, m_2, m_3 and m_4 revolving in planes 1, 2, 3 and 4 respectively as shown in Figure 7 (a). The relative angular positions of these masses are shown in the end view [Figure 7(b)]. The magnitude of the balancing masses m_L and m_M in planes L and M may be obtained as discussed below:

- a) Take one of the planes; say L as the reference plane (R.P.). The distances of all the other planes to the left of the reference plane may be regarded as *negative*, and those to the right as *positive*.
- b) Tabulate the data as shown in Table 1. The planes are tabulated in the same order in which they occur, reading from left to right.



(c) Couple vector. (d) Couple vectors turned (e) Couple polygon. (f) Force polygon. counter clockwise through

a right angle.

Figure 7: Planes and angular Positions of masses

Plane (1)	Mass (m) (2)	Radius(r) (3)	Cent.force $\div \omega^2$ (m.r) (4)	Distance from Plane L (l) (5)	Couple $\div \omega^2$ (m.r.l) (6)
1 L(R.P.) 2 3 M 4	$egin{array}{c} m_1 & m_L & \ m_2 & \ m_3 & \ m_M & \ m_4 & \ \end{array}$	r_1 r_L r_2 r_3 r_M r_4	$m_{1}.r_{1} \\ m_{L}.r_{L} \\ m_{2}.r_{2} \\ m_{3}.r_{3} \\ m_{M}.r_{M} \\ m_{4}.r_{4}$	$-l_1$ 0 l_2 l_3 l_M l_4	$- m_{1} \cdot r_{1} \cdot l_{1} \\ 0 \\ m_{2} \cdot r_{2} \cdot l_{2} \\ m_{3} \cdot r_{3} \cdot l_{3} \\ m_{M} \cdot r_{M} \cdot l_{M} \\ m_{4} \cdot r_{4} \cdot l_{4}$

Table 1: Tabulated planes data

- c) A couple may be represented by a vector drawn perpendicular to the plane of the couple. The couple C_1 introduced by transferring m_1 to the reference plane through O is proportional to m_1 . C_1 . l_1 and acts in a plane through Om_1 and perpendicular to the paper. The vector representing this couple is drawn in the plane of the paper and perpendicular to Om_1 as shown by OC_1 in Figure 7 (c). Similarly, the vectors OC_2 , OC_3 and OC_4 are drawn perpendicular to Om_2 , Om_3 and Om_4 respectively and in the plane of the paper.
- d) The couple vectors as discussed above are turned counter clockwise through a right angle for convenience of drawing as shown in Figure 7 (d). We see that their relative positions remain unaffected. Now the vectors OC_2 , OC_3 and OC_4 are parallel and in the same direction as Om_2 , Om_3 and Om_4 , while the vector OC_1 is parallel to Om_1 but in *opposite direction. Hence the *couple vectors are drawn radially outwards for* the masses on one side of the reference plane and radially inward for the masses on the other side of the reference plane.
- e) Now draw the couple polygon as shown in Figure 7(e). The vector d'o' represents the balanced couple. Since the balanced couple C_M is proportional to $m_M \cdot r_M \cdot l_M$, therefore;

$$C_M = m_M \cdot r_M \cdot l_M = vector \ d'o' \quad \text{or} \ m_M = \frac{vector \ d'o'}{r_M \cdot l_M}$$

From this expression, the value of the balancing mass m_M in the plane M may be obtained, and the angle of inclination ϕ of this mass may be measured from Figure 7 (*b*).

f) Now draw the force polygon as shown in Figure 7(f). The vector eo (in the direction from e to o) represents the balanced force. Since the balanced force is proportional to m_L . r_L , therefore:

$$m_L.r_L = vector\ eo$$
 or $m_L = \frac{vector\ eo}{r_L}$

From this expression, the value of the balancing mass m_L in the plane L may be obtained and the angle of inclination α of this mass with the horizontal may be measured from Figure 7 (b).

- **Example 2:** A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45°, B to C 70° and C to D 120°. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.
- **Solution:** Given : $m_A = 200 \text{ kg}$; $m_B = 300 \text{ kg}$; $m_C = 400 \text{ kg}$; $m_D = 200 \text{ kg}$; $r_A = 80 \text{ mm} = 100 \text{ kg}$ 0.08m; $r_B = 70 \text{ mm} = 0.07 \text{ m}$; $r_C = 60 \text{ mm} = 0.06 \text{ m}$; $r_D = 80 \text{ mm} = 0.08 \text{ m}$; $r_X = r_Y$ = 100 mm = 0.1 m. Let m_X = Balancing mass placed in plane X, and m_Y = Balancing mass placed in plane Y.

The position of planes and angular position of the masses (assuming the mass A as horizontal) are shown in Figure 8 (a) and (b) respectively.

Assume the plane X as the reference plane (R.P.). The distances of the planes to the right of plane X are taken as + ve while the distances of the planes to the left of plane X are taken as – ve. The data may be tabulated as shown in Table 2.

Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	$Cent.force \div \omega^2$ $(m.r) kg-m$ (4)	Distance from Plane x(l) m (5)	Couple $\div \omega^2$ (m.r.l) kg-m ² (6)
A	200	0.08	$ \begin{array}{r} 16 \\ 0.1 \ m_{\rm X} \\ 21 \\ 24 \\ 0.1 \ m_{\rm Y} \\ 16 \\ \end{array} $	- 0.1	-1.6
X(R.P.)	<i>m</i> _X	0.1		0	0
B	300	0.07		0.2	4.2
C	400	0.06		0.3	7.2
Y	<i>m</i> _Y	0.1		0.4	0.04 $m_{\rm Y}$
D	200	0.08		0.6	9.6

Table 2: Tabulated planes data for example number 2

The balancing masses m_X and m_Y and their angular positions may be determined graphically as discussed below:

a) First of all, draw the couple polygon from the data given in Table 2 (column 6) as shown in Figure 8 (c) to some suitable scale. The vector d'o' represents the balanced couple. Since the balanced couple is proportional to 0.04 m_Y, therefore by measurement, 0.04m_Y = vector d'o' = 7.3 kg - m² or m_Y = 182.5 kg



Figure 8: Positions planes and angular positions of masses

The angular position of the mass m_Y is obtained by drawing Om_Y in Figure 8 (*b*), parallel to vector *d'* o'. By measurement, the angular position of m_Y is $\theta_Y = 12^\circ$ in the clockwise direction from mass m_A (*i.e.* 200 kg). **Ans**.

b) Now draw the force polygon from the data given in Table 2 (column 4) as shown in Figure (*d*). The vector *eo* represents the balanced force. Since the balanced force is proportional to $0.1 m_x$, therefore by measurement,

 $0.1m_X =$ Vector eo = 35.5 kg-m or $m_X = 355$ kg Ans.

The angular position of the mass m_X is obtained by drawing Om_X in Figure 8 (*b*), parallel to vector *eo*. By measurement, the angular position of m_X is $\theta_X = 145^\circ$ in the clockwise direction from mass m_A (*i.e.* 200 kg). **Ans**.

Example 3: A, B, C and D are four masses carried by a rotating shaft at radii 100, 125, 200 and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg, and 4 kg respectively. Find the required mass A and the relative angular settings of the four masses so that the shaft shall be in complete balance.

Solution: Given: $r_A = 100 \text{ mm} = 0.1 \text{ m}$; $r_B = 125 \text{ mm} = 0.125 \text{ m}$; $r_C = 200 \text{ mm} = 0.2 \text{ m}$;

 $r_D = 50 \text{ mm} = 0.15 \text{ m}; m_B = 10 \text{ kg}; m_C = 5 \text{ kg}; m_D = 4 \text{ kg}$

The position of planes is shown in Figure 9 (*a*). Assuming the plane of mass A as the reference plane (R.P.), the data may be tabulated as in Table 3:

Table 3: Tabulated planes data for example number 3

Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent. Force $\div \omega^2$ (m.r)kg-m (4)	Distance from plane A (l)m (5)	$Couple \div \omega^2$ (m.r.l) kg-m ² (6)
A(R.P.) B C D	<i>m</i> _A 10 5 4	0.1 0.125 0.2 0.15	$ \begin{array}{c} 0.1 \ m_{\rm A} \\ 1.25 \\ 1 \\ 0.6 \end{array} $	0 0.6 1.2 1.8	0 0.75 1.2 1.08

First of all, the angular setting of masses C and D is obtained by drawing the couple polygon from the data given in Table 3 (column 6). Assume the position of mass B in the

horizontal direction OB as shown in Figure 9 (*b*). Now the couple polygon as shown in Figure 9 (*c*) is drawn as discussed below:

- a) Draw vector o' b' in the horizontal direction (*i.e.* parallel to *OB*) and equal to 0.75 $kg m^2$, to some suitable scale.
- b) From points o' and b', draw vectors o' c' and b'c' equal to $1.2 kg m^2$ and $1.08 kg m^2$ respectively. These vectors intersect at c'.
- c) Now in Figure 9 (b), draw OC parallel to vector o' c' and OD parallel to vector b'c'.
 By measurement, we find that the angular setting of mass C from mass B in the anticlockwise direction, *i.e.*

 $BOC = 240^{\circ}$ Ans.

and angular setting of mass D from mass B in the anticlockwise direction, *i.e.*

 $BOD = 100^{\circ}$ Ans.

In order to find the required mass $A(m_A)$ and its angular setting, draw the force polygon to some suitable scale, as shown in Figure 9 (*d*), from the data given in Table 3 (column 4). Since the closing side of the force polygon (vector *do*) is proportional to 0.1 m_A , therefore; by measurement, 0.1 $m_A = 0.7 kg - m^2$ or $m_A = 7 kg$ **Ans.**

Now draw *OA* in Figure 9 (*b*), parallel to vector *do*. By measurement, we find that the angular setting of mass *A* from mass *B* in the anticlockwise direction, *i.e.*



 $BOA = 155^{\circ}$ Ans.



Figure 9: Positions planes and angular positions of masses

TRY Example 4: Four masses A, B, C and D as shown below are to be completely balanced.

	A	В	С	D
Mass (kg)	_	30	50	40
Radius (mm)	180	240	120	150

The planes containing masses B and C are 300 mm apart. The angle between planes containing B and C is 90°. B and C make angles of 210° and 120° respectively with D in the same sense. Find:

- a) The magnitude and the angular position of mass A; and
- b) The position of planes A and D.

2.1 Balancing of Reciprocating Masses

The resultant of all the forces acting on a reciprocating body of an engine due to inertia forces is usually known as an *unbalanced force* or *shaking force*. Thus if the resultant of all the forces due to inertia effects is zero, then there will be no unbalanced force, but even then an unbalanced couple or shaking couple will be present. Consider a horizontal reciprocating engine mechanism as shown in Figure 10.



Figure 10: Reciprocating Engine Mechanism

Let F_R = Force required to accelerate the reciprocating parts,

 F_I = Inertia force due to reciprocating parts,

 F_N = Force on the sides of the cylinder walls or normal force acting on the cross-head guides, and

 F_B = Force acting on the crankshaft bearing or main bearing.

Since F_R and F_I are equal in magnitude but opposite in direction, therefore they balance each other. The horizontal component of F_B (*i.e.* F_{BH}) acting along the line of reciprocation is also equal and opposite to F_I . This force $F_{BH} = F_U$ is an unbalanced force or shaking force and required to be properly balanced. The force on the sides of the cylinder walls (F_N) and the vertical component of F_B (*i.e.* F_{BV}) are equal and opposite and thus form a shaking couple of magnitude $F_N \times x$ or $F_{BV} \times x$.

From above we see that the effect of the reciprocating parts is to produce a shaking force and a shaking couple. Since the shaking force and a shaking couple vary in magnitude and direction during the engine cycle, therefore they cause very objectionable vibrations. Thus the purpose of balancing the reciprocating masses is to eliminate the shaking force and a shaking couple. In most of the mechanisms, we can reduce the shaking force and a shaking couple by adding appropriate balancing mass, but it is usually not practical to eliminate them completely. In other words, the reciprocating masses are only partially balanced.

Note: The masses rotating with the crankshaft are normally balanced and they do not transmit any unbalanced or shaking force on the body of the engine.

2.1.1 Primary and Secondary Unbalanced Forces of Reciprocating Masses

Consider a reciprocating engine mechanism as shown in Figure 10.

Let m = Mass of the reciprocating parts,

l = Length of the connecting rod PC,

r =Radius of the crank OC,

 θ = Angle of inclination of the crank with the line of stroke *PO*,

 ω = Angular speed of the crank,

n =Ratio of length of the connecting rod to the crank radius = l / r.

The acceleration of the reciprocating parts is approximately given by the expression,

 $a_R = \omega^2 \cdot r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$

 \therefore Inertia force due to reciprocating parts or force required to accelerate the reciprocating parts,

$$F_1 = F_R = Mass \times Acceleration = m. \omega^2. r\left(cos\theta + \frac{cos2\theta}{n}\right)$$

We have discussed in the previous article that the horizontal component of the force exerted on the crank shaft bearing (*i.e.* F_{BH}) is equal and opposite to inertia force (F_I). This force is an unbalanced one and is denoted by F_U

 \therefore Unbalanced force,

$$F_U = m.\,\omega^2.\,r\left(\cos\theta + \frac{\cos2\theta}{n}\right) = m.\,\omega^2.\,r\cos\theta + m.\,\omega^2.\,r\frac{\cos2\theta}{n} = F_P + F_S$$

The expression $m. \omega^2. rcos\theta$ is known as *primary unbalanced force* and $m. \omega^2. r\frac{cos2\theta}{n}$ is called *secondary unbalanced force*.

Primary unbalanced force, $F_P = m. \omega^2. rcos\theta$ and secondary unbalanced force,

$$F_s = m.\,\omega^2.\,r\,\frac{\cos 2\theta}{n}$$

~ ~

- Notes 1: The primary unbalanced force is maximum, when $\theta = 0^{\circ}$ or 180°. Thus, the primary force is maximum twice in one revolution of the crank. The maximum primary unbalanced force is given by $F_{P_{max}} = m. \omega^2. r.$
 - 2. The secondary unbalanced force is maximum, when $\theta = 0^{\circ}, 90^{\circ}, 180^{\circ}$ and 360°. Thus, the secondary force is maximum four times in one revolution of the crank. The maximum secondary unbalanced force is given by $F_{s_{max}} = m. \omega^2 \cdot \frac{r}{n}$.
 - 3. From above we see that secondary unbalanced force is 1/n times the maximum primary unbalanced force.
 - 4. In case of moderate speeds, the secondary unbalanced force is so small that it may be neglected as compared to primary unbalanced force.
 - 5. The unbalanced force due to reciprocating masses varies in magnitude but constant in direction while due to the revolving masses; the unbalanced force is constant in magnitude but varies in direction.

Partial Balancing of Unbalanced Primary Force in a Reciprocating Engine 2.1.2

The primary unbalanced force $F_P = m. \omega^2 . rcos\theta$ may be considered as the component of the centrifugal force produced by a rotating mass m placed at the crank radius r, as shown in Figure 11.



Figure 11: Partial balancing of unbalanced primary force in a reciprocating engine

The primary force acts from O to P along the line of stroke. Hence, balancing of primary force is considered as equivalent to the balancing of mass m rotating at the crank radius r. This is balanced by having a mass B at a radius b, placed diametrically opposite to the crank pin C.

We know that centrifugal force due to mass *B*,

$$= B. \omega^2. b$$

and horizontal component of this force acting in opposite direction of primary force;

$$= B. \omega^2. bcos\theta$$

The primary force is balanced, if

$$B. \omega^2. bcos\theta = m. \omega^2. rcos\theta$$
 or $B.b = m.r$

A little consideration will show that the primary force is completely balanced if B.b =m.r but the centrifugal force produced due to the revolving mass B, has also a vertical component (perpendicular to the line of stroke) of magnitude $B.\omega^2.bsin\theta$. This force remains unbalanced. The maximum value of this force is equal to $B.\omega^2.b$ when θ is 90° and 270°, which is same as the maximum value of the primary force $m.\omega^2.r$.

From the above discussion, we see that in the first case, the primary unbalanced force acts along the line of stroke whereas in the second case, the unbalanced force acts along the perpendicular to the line of stroke. The maximum value of the force remains same in both the cases. It is thus obvious, that the effect of the above method of balancing is to change the direction of the maximum unbalanced force from the line of stroke to the perpendicular of line of stroke. As a compromise let a fraction 'c' of the reciprocating masses is balanced, such that

c.m.r = B.b

Unbalanced force along the line of stroke

$$= m. \omega^{2}.rcos\theta - B. \omega^{2}.bcos\theta$$
$$= m. \omega^{2}.rcos\theta - c.m. \omega^{2}.rcos\theta$$
$$= (1 - c)m. \omega^{2}.rcos\theta$$

and unbalanced force along the perpendicular to the line of stroke

$$= B.\omega^2.bsin\theta = c.m.\omega^2.rsin\theta$$

Resultant unbalanced force at any instant ...

$$= \sqrt{[(1-c)m.\omega^2.rcos\theta]^2 + [c.m.\omega^2.rsin\theta]^2}$$
$$= m.\omega^2.r\sqrt{(1-c)^2cos^2\theta + c^2.sin^2\theta}$$

Note: If the balancing mass is required to balance the revolving masses as well as reciprocating masses, then $B.b = m_1.r_1 + c.m.r = (m_1 + c.m)r$ Where m_1 = Magnitude of the revolving masses, and m = magnitude of the reciprocating masses

Example 4: A single cylinder reciprocating engine has speed 240 rpm, stroke 300 mm, mass of reciprocating parts 50 kg, mass of revolving parts at 150 mm radius 37 kg. If two third of the reciprocating parts and all the revolving parts are to be balanced, find:

- *i.* The balance mass required at a radius of 400 mm, and
- *ii.* The residual unbalanced force when the crank has rotated 60° from top dead center.

Solution: Given; N = 240 r. p. m. or $\omega = 2\pi \times 240 / 60 = 25.14 \text{ rad/s}$;

Stroke = $300 mm = 0.3 m; m = 50 kg; m_1 = 37 kg;$

r = 150 mm = 0.15 m; c = 2/3

1. Balance mass required

Let B = Balance mass required, and

b =Radius of rotation of the balance mass = 400 mm = 0.4 m ... (Given) We know that

$$B.b = (m_1 + c.m) r$$

$$B \times 0.4 = \left(37 \times \frac{2}{3}50\right) 0.15 = 10.55 \text{ or } B = 26.38 kg$$

2. Residual unbalanced force

Let θ = Crank angle from top dead centre = 60° ... (Given)

We know that residual unbalanced force

$$= m. \omega^{2} \cdot r \sqrt{(1-c)^{2} \cos^{2}\theta + c^{2} \cdot \sin^{2}\theta}$$
$$= 50. (25.14)^{2} \cdot 0.15 \sqrt{(1-\frac{2}{3})^{2} \cos^{2}60 + \left(\frac{2}{3}\right)^{2} \cdot \sin^{2}60}$$
$$= 4740 \times 0.601 = 2849N$$

2.1.3 Partial Balancing of Locomotives

The locomotives, usually, have two cylinders with cranks placed at right angles to each other in order to have uniformity in turning moment diagram. The two cylinder locomotives may be classified as:

- i. Inside cylinder locomotives; and
- ii. Outside cylinder locomotives.

In the *inside cylinder locomotives*, the two cylinders are placed in between the planes of two driving wheels as shown in Figure 12(a); whereas in the *outside cylinder locomotives*, the two cylinders are placed outside the driving wheels, one on each side of the driving wheel, as shown in Figure 12(b). The locomotives may be

(a) Single or uncoupled locomotives; and (b) Coupled locomotives.



Figure 12: Inside and Outside Locomotive Cylinder

A *single* or *uncoupled locomotive* is one, in which the effort is transmitted to one pair of the wheels only; whereas in *coupled locomotives*, the driving wheels are connected to the leading and trailing wheel by an outside coupling rod.

2.1.4 Effect of Partial Balancing of Reciprocating Parts of Two Cylinder Locomotives

We have discussed in the previous article that the reciprocating parts are only partially balanced. Due to this partial balancing of the reciprocating parts, there is an unbalanced primary force along the line of stroke and also an unbalanced primary force perpendicular to the line of stroke. The effect of an unbalanced primary force along the line of stroke is to produce:

- i. Variation in tractive force along the line of stroke; and
- ii. Swaying couple.

The effect of an unbalanced primary force perpendicular to the line of stroke is to produce variation in pressure on the rails, which results in hammering action on the rails. The maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as a *hammer blow*. We shall now discuss the effects of an unbalanced primary force in the following articles.

2.1.4.1 Variation of Tractive Force

The resultant unbalanced force due to the two cylinders, along the line of stroke, is known as *tractive force*. Let the crank for the first cylinder be inclined at an angle θ with the line of stroke, as shown in Figure 13. Since the crank for the second cylinder is at right angle to the first crank, therefore the angle of inclination for the second crank will be $(90^{\circ} + \theta)$.



Figure 13: Variation of tractive force

Let m = Mass of the reciprocating parts per cylinder, and

c = Fraction of the reciprocating parts to be balanced.

We know that unbalanced force along the line of stroke for cylinder 1

 $= (1-c)m.\omega^2.rcos\theta$

Similarly, unbalanced force along the line of stroke for cylinder 2,

$$= (1-c)m.\omega^2.rcos(90^o+\theta)$$

 \therefore As per definition, the tractive force,

 F_T = Resultant unbalanced force along the line of stroke

$$= (1 - c)m.\omega^{2}.rcos\theta + (1 - c)m.\omega^{2}.rcos(90^{o} + \theta)$$
$$= (1 - c)m.\omega^{2}.r(cos\theta - sin\theta)$$

The tractive force is maximum or minimum when $(\cos\theta - \sin\theta)$ is maximum or minimum.

For $(cos\theta - sin\theta)$ to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta - \sin\theta) = 0 \quad or \quad -\sin\theta - \cos\theta = 0 \quad or \quad -\sin\theta = \cos\theta$$

$$\therefore \quad \tan\theta = -1 \quad or \quad \theta = 135^{\circ} \text{ or } 315^{\circ}$$

Thus, the tractive force is maximum or minimum when $\theta = 135^{\circ}$ or 315° .

Maximum and minimum value of the tractive force or the variation in tractive force

$$= \pm (1-c)m.\omega^2.r(\cos 135^\circ - \sin 315^\circ) = \pm \sqrt{2}(1-c)m.\omega^2.r$$

2.1.4.2 Swaying couple

The unbalanced forces along the line of stroke for the two cylinders constitute a couple about the center line YY between the cylinders as shown in Figure 14.



Figure 14: Swaying couple.

This couple has swaying effect about a vertical axis, and tends to sway the engine alternately in clockwise and anticlockwise directions. Hence the couple is known as swaying couple.

Let *a* = Distance between the center lines of the two cylinders.

Swaying couple = $(1 - c)m.\omega^2.rcos\theta \times \frac{a}{2} - (1 - c)m.\omega^2.rcos(90^o + \theta) \times \frac{a}{2}$ $= (1-c)m.\omega^2.rcos\theta \times \frac{a}{2}(cos\theta - sin\theta)$

The swaying couple is maximum or minimum when $(\cos\theta - \sin\theta)$ is maximum or minimum.

For $(cos\theta + sin\theta)$ to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta + \sin\theta) = 0 \quad or \quad -\sin\theta + \cos\theta = 0 \quad or \quad -\sin\theta = -\cos\theta$$
$$\tan\theta = 1 \quad or \quad \theta = 45^{\circ} \text{ or } 225^{\circ}$$

Thus, the swaying couple is maximum or minimum when $\theta = 45^{\circ}$ or 225°.

Maximum and minimum value of the tractive force or the variation in swaying couple

$$= \pm (1-c)m.\,\omega^2.r \times \frac{a}{2}(\cos 45^o + \sin 45^o) = \pm \frac{a}{\sqrt{2}}(1-c)m.\,\omega^2.n$$

2.1.4.3 Hammer Blow

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We have already discussed that the maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as *hammer blow*.

We know that the unbalanced force along the perpendicular to the line of stroke due to the balancing mass B, at a radius b, in order to balance reciprocating parts only is $B. \omega^2. bsin\theta$. This force will be maximum when $sin\theta$ is unity, *i.e.* when $\theta = 90^{\circ} \text{ or } 270^{\circ}.$

Hammer blow = $B. \omega^2. b$ (Substituting $sin\theta = 1$) :.

The effect of hammer blow is to cause the variation in pressure between the wheel and the rail. This variation is shown in Figure 15, for one revolution of the wheel.

Let *P* be the downward pressure on the rails (or static wheel load).

Net pressure between the wheel and the rail :.

$$= P \pm B. \omega^2. b$$



Figure 15: Hammer blow

If $(P - B.\omega^2.b)$ is *negative*, then the wheel will be lifted from the rails; therefore the limiting condition in order that the wheel does not lift from the rails is given by P = $B. \omega^2. b$ and the permissible value of the angular speed, $\omega = \sqrt{\frac{p}{B.b}}$

- **Example 5:** The following data refer to two cylinder locomotive with cranks at 90°: Reciprocating mass per cylinder = 300kg; Crank radius = 0.3 m; Driving wheel diameter = 1.8m; Distance between cylinder center lines = 0.65m; Distance between the driving wheel central planes = 1.55m. Determine:
 - i. The fraction of the reciprocating masses to be balanced, if the hammer blow is not to exceed 46kN at 96.5 km. ph.;
 - The variation in tractive effort; and ii.
 - iii. The maximum swaying couple.

Solution: Given m = 300 kg; r = 0.3 m; D = 1.8 m or R = 0.9 m; a = 0.65 m; Hammer blow = $46 kN = 46 \times 103 N$; v = 96.5 km/h = 26.8 m/s

i. Fraction of the reciprocating masses to be balanced

Let c = Fraction of the reciprocating masses to be balanced, and

B = Magnitude of balancing mass placed at each of the driving wheels at radius b.

We know that the mass of the reciprocating parts to be balanced = c.m. = 300c kg

The position of planes of the wheels and cylinders is shown in Figure 16(a), and the position of cranks is shown in Figure 16(b). Assuming the plane of wheel A as the reference plane, the data may be tabulated as below:





Table 5: Question 5 Data

Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent. force $\div \omega^2$ (m.r) kg-m (4)	Distance from plane A (l)m (5)	Couple $\div \omega^2$ (m.r.l.) kg-m ² (6)
A (R.P.)	В 300 с	<i>b</i>	<i>B.b</i>	0	0 40.5 c
C D	300 c B	0.3 b	90 c B.b	1.1 1.55	99 c 1.55 B.b

Now the couple polygon, to some suitable scale, may be drawn with the data given in Table 5 (column 6), as shown in Figure 16b. The closing side of the polygon (vector c' o') represents the balancing couple and is proportional to 1.55 *B*. *b*.



Figure 16b: couple polygon

From the couple polygon,

1.55B. b =
$$\sqrt{(40.5c)^2 + (99c)^2} = 107c$$

∴ B. b = $\frac{107c}{1.55} = 69c$

We know that angular speed $\omega = \frac{v}{R} = \frac{26.8}{0.9} = 29.8 rad/s$

Hammer blow $46 \times 10^3 = B. \omega^2. b$...

$$46 \times 10^3 = 69c(29.8)^2 = 61275c$$

 $c = 46 \times 10^3/61275 = 0.751$

ii. Variation in tractive effort

We know that variation in tractive effort

$$= \pm \sqrt{2}. (1 - c)m. \omega^2. r = \pm \sqrt{2}. (1 - 0.751)300(29.8)^2. 0.3$$
$$= 28140 = 28.14kN$$

Maximum Swaying couple iii.

$$=\frac{a(1-c)}{\sqrt{2}}.m.\omega^{2}.r=\frac{0.65(1-0.751)}{\sqrt{2}}.300(29.8)^{2}.0.3=9148\,N-m=9.148kN-m$$

Example 6: An inside cylinder locomotive has its cylinder center lines 0.7 m apart and has a stroke of 0.6 m. The rotating masses per cylinder are equivalent to 150 kg at the crank pin, and the reciprocating masses per cylinder are equivalent to 180 kg. The wheel center lines are 1.5 m apart. The cranks are at right angles. The whole of the rotating and 2/3 of the reciprocating masses are to be balanced by masses placed at a radius of 0.6 m. Find the magnitude and direction of the balancing masses. Find the fluctuation in rail pressure under one wheel, variation of tractive effort and the magnitude of swaying couple at a crank speed of 300 rpm.

Solution: Given a = 0.7 m; $l_B = l_C = 0.6 m$ or $r_B = r_C = 0.3 m$; $m_1 = 150 kg$;

$$m_2 = 180 \text{ kg}; \ c = 2/3; r_A = r_D = 0.6 \text{ m}; N = 300 \text{ r. p. m. or}$$

 $\omega = 2\pi \times 300 / 60 = 31.42 \text{ rad/s}$

We know that the equivalent mass of the rotating parts to be balanced per cylinder at the crank pin, $m = m_B = m_C = m_1 + c. m_2 = 150 + \frac{2}{3}(180) = 270 kg$

Magnitude and direction of the balancing masses

Let m_A and m_D = Magnitude of the balancing masses

 θ_A and θ_D = Angular position of the balancing masses m_A and m_D from the first crank *B*. The magnitude and direction of the balancing masses may be determined graphically as discussed below:

- i. First of all, draw the space diagram to show the positions of the planes of the wheels and the cylinders, as shown in Figure 17(a). Since the cranks of the cylinders are at right angles, therefore assuming the position of crank of the cylinder *B* in the horizontal direction, draw *OC* and *OB* at right angles to each other as shown in Figure 17(b).
- ii. Tabulate the data as given in the following table. Assume the plane of wheel *A* as the reference plane.



Figure 17: Planes and Angular positions of the wheels and cylinder

Table 6:	Question	6	Data
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Plane (1)	mass. (m) kg (2)	Radius (r)m (3)	Cent. force $\div \omega^2$ (m.r) kg-m (4)	Distance from plane A (l)m (5)	$\begin{array}{c} Couple \div \omega^2 \\ (m.r.l) \ kg-m^2 \\ (6) \end{array}$
A (R.P.)	m _A	0.6	0.6 m _A	0	0
В	270	0.3	81	0.4	32.4
С	270	0.3	81	1.1	89.1
D	m _D	0.6	0.6m _D	1.5	0.9 m _D

iii. Now, draw the couple polygon from the data given in Table 6 (column 6), to some suitable scale, as shown in Figure 18(c). The closing side c' o' represents the balancing couple and it is proportional to 0.9 m_D . Therefore, by measurement,

 $0.9 m_D = vector c' o' = 94.5 kg - m^2 or m_D = 105 kg$ Ans.



iv. To determine the angular position of the balancing mass *D*, draw *OD* in Figure 17(*b*) parallel to vector c' o'. By measurement, $\theta_D = 250^\circ$ Ans.

In order to find the balancing mass A, draw the force polygon from the data given v. in Table 6 (column 4), to some suitable scale, as shown in Figure 18(d), The vector do represents the balancing force and it is proportional to $0.6 m_A$. Therefore by measurement,

 $0.6 m_A = vector do = 63 kg - m or m_A = 105 kg Ans.$

To determine the angular position of the balancing mass A, draw OA in Fig. 18 (b) vi. parallel to vector do. By measurement,

 $\theta_A = 200^\circ$ Ans.

Fluctuation in rail pressure

We know that each balancing mass = 105 kg

:. Balancing mass for rotating masses,

$$D = \frac{m_1}{m} \times 105 = \frac{150}{270} \times 105 = 58.3 kg$$

and balancing mass for reciprocating masses,

$$B = \frac{c.m_2}{m} \times 105 = \frac{2}{3} \times \frac{180}{270} \times 105 = 46.6kg$$

This balancing mass of 46.6 kg for reciprocating masses gives rise to the centrifugal force.

Fluctuation in rail pressure or hammer blow

$$= B \cdot \omega^2 \cdot b = 46.6 (31.42)^2 0.6 = 27\ 602\ N \cdot Ans. \qquad \dots (\because b = r_A = r_D)$$

Variation of tractive effort

We know that maximum variation of tractive effort

$$= \pm \sqrt{2}. (1-c)m. \omega^2. r == \pm \sqrt{2}. \left(1 - \frac{2}{3}\right) 180. (31.42)^2. 0.3 N$$

 $= \pm 25 \ 127 \ N$ **Ans**.

Swaying couple

We know that maximum swaying couple,

$$= \frac{a(1-c)}{\sqrt{2}} \cdot m \cdot \omega^2 \cdot r = \frac{0.7\left(1-\frac{2}{3}\right)}{\sqrt{2}} \cdot 180 \cdot (31.42)^2 \cdot 0.3$$
$$= 8797 N - m Ans.$$

3.0 **VIBRATIONS**

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a vibratory motion. This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position. When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilibrium position. In this way, the vibratory motion is repeated indefinitely. Thus the idea of theory of vibration is to study the behavior of this motion and develop mathematical model equations that would control the effects of this motion to a desired level.

3.1 Terms Used in Vibratory Motions

The following terms are commonly used in connection with the vibratory motions:

Period of vibration or time period: It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

Cycle: It is the motion completed during one time period.

Frequency: It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

3.2 Types of Vibratory Motion

The following types of vibratory motion are important from the subject point of view:

i. **Free or natural vibrations:** When no external force acts on the body, after giving it an initial displacement, then the body is said to be under free or natural vibrations. The frequency of the free vibrations is called free or natural frequency.

- ii. **Forced vibrations:** When the body vibrates under the influence of external force, then the body is said to be under forced vibrations. The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.
- iii. **Damped vibrations:** When there is a reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration. This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

Note: When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

3.3 Types of Free Vibrations

The following three types of free vibrations are important from the subject point of view:

- i. Longitudinal vibrations: When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig.19 (a), then the vibrations are known as longitudinal vibrations. In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.
- ii. **Transverse vibrations:** When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. 19 (b), then the vibrations are known as transverse vibrations. In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft and
- iii. Torsional vibrations: When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. 19 (c), then the vibrations are known as torsional vibrations. In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

Note: If the limit of proportionality (i.e. stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, simple harmonic.



B = Mean position ; A and C = Extreme positions. (a) Longitudinal vibrations. (b) Transverse vibrations. (c) Torsional vibrations.

Figure 19: Types of free vibrations

3.4 Natural Frequency of Free Longitudinal Vibrations

The natural frequency of the free longitudinal vibrations may be determined by Equilibrium Method:

3.4.1 Equilibrium Method

Consider a constraint (i.e. spring) of negligible mass in an unstrained position, as shown in Fig. 2 (a).

Let s = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m (NB: s = k).

m = Mass of the body suspended from the constraint in kg,

W = Weight of the body in Newton = m.g

- δ = Static deflection of the spring in meters due to weight W in Newton, and
- x = Displacement given to the body by the external force, in meters.



Figure 20: Natural frequency of free longitudinal vibrations

In the equilibrium position, as shown in Figure 20 (b), the gravitational pull W = m.g is balanced by a force of spring, such that $W = s. \delta$ -----1 Since the mass is now displaced from its equilibrium position by a distance x, as shown in Figure 20 (c), and is then released, therefore after time t,

Restoring force = $W - s(\delta + x) = W - s \cdot \delta - s \cdot x$

This is the same thing as = s. δ – s. δ – s. x = –s. x . . (Taking upward force as negative)

Accelerating force = Mass × Acceleration = M × $\frac{d^2t}{dx^2}$ ------2

(Taking downward force as positive)

Equating equations 1 and 2, the equation of motion of the body of mass m after time t is

We know that the fundamental equation of simple harmonic motion is

Comparing equations 3 and 4 we have

$$\omega = \sqrt{\frac{s}{m}}$$

... Time period,

$$t_p = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{s}} \qquad ----5$$

and natural frequency,

$$f_n = \frac{1}{t_p} = \frac{1}{2\pi} \sqrt{\frac{s}{m}} = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} \qquad \therefore (m.g = s.\delta) \qquad -----6$$

Taking the value of g as 9.81 m/s2 and δ in meters,

$$f_n = \frac{1}{2\pi} \sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}} Hz$$

Note: The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = \text{E} \quad \text{or} \quad \frac{\text{w}}{\text{A}} \times \frac{1}{\delta} = \text{E} \quad \text{or} \quad \delta = \frac{\text{W.I}}{\text{E.A}}$$

Where δ = Static deflection i.e. extension or compression of the constraint,

W = Load attached to the free end of constraint,

l = Length of the constraint,

E = Young's modulus for the constraint, and

A = Cross-sectional area of the constraint.

3.5 Natural Frequency of Free Transverse Vibrations

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight *W*, as shown in the Figure 21 below.



Figure 21: A shaft of negligible mass

Let s =Stiffness of shaft,

 δ = Static deflection due to weight of the body,

x = Displacement of body from mean position after time t.

m = Mass of body = W/g

As discussed above

Restoring force = -s.x....(i)

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and accelerating force= $M \times \frac{d^2x}{dt^2}$ (ii)

Equating equations (*i*) and (*ii*), the equation of motion becomes Thus;

$$M \times \frac{d^2 x}{dt^2} = -s. x \quad \text{or } M \times \frac{d^2 x}{dt^2} + s. x = 0$$
$$\frac{d^2 x}{dt^2} + \frac{s}{m} x = 0 \qquad \text{Same as before}$$

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

Time period,
$$t_p = 2\pi \sqrt{\frac{m}{s}}$$

and natural frequency, $f_n = \frac{1}{t_n} = \frac{1}{2\pi} \sqrt{\frac{s}{m}} = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}}$

Note: The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on Strength of Materials that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI}$$

Where

...

l = Length of the shaft or beam in meters,

W = Load at the free end, in Newton,

E = Young's modulus for the material of the shaft or beam in N/m², and

I = Moment of inertia of the shaft or beam in m⁴.

Example 7: A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 GN/m². Determine the frequency of longitudinal and transverse vibrations of the shaft.

Solution: Given: d = 50 mm = 0.05m; l = 300 mm = 0.03m; m = 100 kg;

 $E = 200 \; GN/m^2 = 200 \times 10^9 \; N/m^2$

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} (0.05)^2 = 1.96 \times 10^{-3} m^2$$

and moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} (0.05)^4 = 0.3 \times 10^{-6} m^4$$

Frequency of longitudinal vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l}{A.E} = \frac{100 \times 9.82 \times 0.3}{1.96 \times 10^{-3} \times 200 \times 10^{9}} = 0.751 \times 10^{-6} m$$

: Frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.751 \times 10^{-6}}} = 575 \ Hz$$
 Ans

Frequency of transverse vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l^3}{3 \times E.I} = \frac{100 \times 9.82 \times (0.3)^3}{3 \times 200 \times 10^9 \times 0.3 \times 10^{-6}} = 0.147 \times 10^{-3} m$$
$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.147 \times 10^{-3}}} = 41 Hz \qquad \text{Ans}$$

3.6.1 Natural Frequency of Free Transverse Vibrations Due to a Point Load Acting Over a Simply Supported Shaft

Consider a shaft *AB* of length *l*, carrying a point load *W* at *C* which is at a distance of l_1 from *A* and l_2 from *B*, as shown in figure below. A little consideration will show that when the shaft is deflected and suddenly released, it will make transverse vibrations. The deflection of the shaft is proportional to the load *W* and if the beam is deflected beyond the static equilibrium position then the load will vibrate with simple harmonic motion (as by a helical spring). If δ is the static deflection due to load *W*, then the natural frequency of the free transverse vibration is

$$f_n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} Hz$$

Figure 22: Simply Supported Beam with a Point Load

Some of the values of the static deflection for the various types of beams and under various load conditions are given in the following table.

Table 7. Values of static deflection (δ) for the various types of beams and under various load conditions

S.No.	Type of beam	$Deflection$ (δ)
1.	Cantilever beam with a point load W at the free end.	$\delta = \frac{Wl^3}{3EI}$ (at the free end)
2.	Cantilever beam with a uniformly distributed load of w per unit length.	$\delta = \frac{wl^4}{8EI}$ (at the free end)
3.	Simply supported beam with an eccentric point load W. W \downarrow \downarrow $a \rightarrow 4$ $b \rightarrow 1$	$\delta = \frac{Wa^2b^2}{3EIl} $ (at the point load)
4.	Simply supported beam with a central point load W. W \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow	$\delta = \frac{Wl^3}{48EI} (\text{at the centre})$

5.	Simply supported beam with a uniformly distributed load of w per unit length.	$\delta = \frac{5}{384} \times \frac{wl^4}{EI} \text{ (at the centre)}$
6.	Fixed beam with an eccentric point load W .	$\delta = \frac{Wa^3b^3}{3E I l} \text{ (at the point load)}$
7.	Fixed beam with a central point load W . W $W\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$	$\delta = \frac{Wl^3}{192EI} (\text{at the centre})$
8.	Fixed beam with a uniformly distributed load of w per unit length.	$\delta = \frac{wl^4}{384EI} \text{ (at the centre)}$

Example 8: A shaft of length 0.75 m, supported freely at the ends, is carrying a body of mass 90 kg at 0.25 m from one end. Find the natural frequency of transverse vibration. Assume E = 200 GN/m^2 and shaft diameter = 50 mm.

Solution: Given l = 0.75 m; m = 90kg; a = AC = 0.25m; E = 200GN/m² = 200×10^{9} N/m²; d = 50mm = 0.05m

The shaft is shown in the figure 23 below;



Figure 23: Simply Supported Beam with a Point Load

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} (0.05)^4 m^4$$
$$= 0.307 \times 10^{-6} m^4$$

and static deflection at the load point (*i.e.* at point *C*),

$$\delta = \frac{Wa^2b^2}{3EIl} = \frac{90 \times 9.81 \times (0.25)^2 (0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 0.75}$$
$$= 0.1 \times 10^{-3}m$$

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.1 \times 10^{-3}}} = 49.85 Hz$$

Example 9: A flywheel is mounted on a vertical shaft as shown in the figure below. The both ends of the shaft are fixed and its diameter is 50 mm. The flywheel has a mass of 500 kg. Find the natural frequencies of longitudinal and transverse vibrations. Take E = 200 GN/m^2 .



Figure 24: A flywheel on a vertical shaft

Solution: Given: d = 50mm = 0.05m; m = 500kg; $E = 200GN/m^2 = 200 \times 10^9 N/m^2$

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} (0.05)^2 = 1.96 \times 10^{-3} m^2$$

and moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} (0.05)^4 = 0.307 \times 10^{-6} m^4$$

Natural frequency of longitudinal vibration

Let m_1 = Mass of flywheel carried by the length l_1 .

 $\therefore m - m_1 =$ Mass of flywheel carried by length l_2 .

We know that extension of length l_1

$$=\frac{W_1*l_1}{A*E} = \frac{m_1g \, l_1}{A*E}$$
(i)

Similarly, compression of length l_2

$$=\frac{(W-W_1)*l_2}{A*E} = \frac{(m-m_1)g\,l_2}{A*E}$$
(ii)

Since extension of length l_1 must be equal to compression of length l_2 , therefore equating equations (*i*) and (*ii*)

$$m_1 l_1 = (m - m_1) l_2$$

$$m_1 \times 0.9 = (500 - m_1) 0.6 = 300 - 0.6m_1 \text{ or } m_1 = 200 kg$$

 \therefore Extension of length l_1 ,

$$\delta = \frac{m_{1*g} l_1}{A*E} = \frac{200 \times 9.81 \times 0.9}{1.96 \times 10^{-3} \times 200 \times 10^9} = 4.5 \times 10^{-6} m$$

We know that natural frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{4.5 \times 10^{-6}}} = 235 Hz$$

Natural frequency of transverse vibration

We know that the static deflection for a shaft fixed at both ends and carrying a point load is given by;

$$\delta = \frac{Wa^3b^3}{3 E \, ll^3} = \frac{500 \times 9.81 \times (0.9)^3 (0.6)^3}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} (1.5)^3} = 1.24 \times 10^{-3} m$$

... (Substituting W = mg, $a = l_1$, and $b = l_2$)

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{1.24 \times 10^{-3}}} = 14.24 Hz$$

4.0 CRITICAL OR WHIRLING SPEED OF A SHAFT

In actual practice, a rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc. When the gears or pulleys are put on the shaft, the center of gravity of the pulley or gear does not coincide with the center line of the bearings or with the axis of the shaft, when the shaft is stationary. This means that the center of gravity of the pulley or gear is at a certain distance from the axis of rotation and due to this, the shaft is subjected to centrifugal force. This force will bent the shaft which will further increase the distance of center of gravity of the pulley or gear from the axis of rotation. This correspondingly increases the value of centrifugal force, which further increases the distance of center of gravity from the axis of rotation. This effect is cumulative and ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity (distance between center of gravity of the pulley and the axis of rotation) but also depends upon the speed at which the shaft rotates. The speed, at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as *critical* or *whirling speed*.



Figure 25: Whirling Speed of a Shaft

Consider a shaft of negligible mass carrying a rotor, as shown in Figure (*a*). The point o is on the shaft axis and G is the center of gravity of the rotor. When the shaft is stationary, the center line of the bearing and the axis of the shaft coincide. Figure (*b*) shows the shaft when rotating about the axis of rotation at a uniform speed of $\omega rad/s$.

Let m = Mass of the rotor,

- e = Initial distance of center of gravity of the rotor from the center line of the bearing or shaft axis, when the shaft is stationary,
- y = Additional deflection of center of gravity of the rotor when the shaft starts rotating at $\omega rad/s$, and

s = Stiffness of the shaft *i.e.* the load required per unit deflection of the shaft.

Since the shaft is rotating at $\omega rad/s$, therefore centrifugal force acting radially outwards through G causing the shaft to deflect is given by

$$F_c = m.\,\omega^2(y+e)$$

The shaft behaves like a spring. Therefore the force resisting the deflection *y*,

$$= s.y$$

For the equilibrium position,

$$m. \omega^{2}(y+e) = s. y$$

or $m\omega^{2}. y + m\omega^{2}. e = s. y$ or $y(s - m. \omega^{2}) = m\omega^{2}. e$
$$y = \frac{m\omega^{2}.e}{s - m.\omega^{2}} = \frac{\omega^{2}.e}{s/m - \omega^{2}}$$

...

We know that circular frequency

$$\omega_n = \sqrt{\frac{s}{m}} \text{ or } y = \frac{\omega^2 \cdot e}{(\omega_n)^2 - \omega^2}$$

A little consideration will show that when $\omega > \omega_n$, the value of y will be negative and the shaft deflects in the opposite direction as shown dotted in Figure (b). In order to have the value of y always positive, both *plus* and *minus* signs are taken.

$$\therefore \qquad y = \pm \frac{\omega^2 \cdot e}{(\omega_n)^2 - \omega^2} = \pm \frac{e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \pm \frac{e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} \qquad \text{(Substituting } \omega_n = \omega_c\text{)}$$

We see from the above expression that when $\omega_n = \omega_c$, the value of y becomes infinite. Therefore ω_c is the **critical or whirling speed.**

:. Critical or whirling speed,

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} Hz$$
 $\left(\delta = \frac{m \cdot g}{s}\right)$

If N_c is the critical or whirling speed in r.p.s., then

$$\therefore \qquad 2\pi N_c = \sqrt{\frac{g}{\delta}} \text{ or } N_c = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ rps}$$

Where δ = critical deflection of the shaft in meters.

Hence, the critical whirling of speed is the same as the natural frequency of transverse vibration but its unit will be revolution per second.

Notes:

- 1. When the center of gravity of the rotor lies between the center line of the shaft and the center line of the bearing, e is taken negative. On the other hand, if the center of gravity of the rotor does not lie between the center line of the shaft and the center line of the bearing (as in the above article) the value of *e* is taken positive.
- 2. To determine the critical speed of a shaft which may be subjected to point loads, uniformly distributed load or combination of both, find the frequency of transverse vibration which is equal to critical speed of a shaft in r.p.s.
- 3. A shaft supported in short bearings (or ball bearings) is assumed to be a simply supported shaft while the shaft supported in long bearings (or journal bearings) is assumed to have both ends fixed.
- Examples10: Calculate the whirling speed of a shaft 20 mm diameter and 0.6 m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m³, and Young's modulus is 200 GN/m^2 . Assume the shaft to be freely supported.

Solution: Given d = 20 mm = 0.02 m; l = 0.6 m; $m_1 = 1 kg$;

 $\rho = 40 Mg/m^3 = 40 \times 10^3 \text{ kg/m}^3$; $E = 200 GN/m^2 = 200 \times 10^9 N/m^2$

The shaft is shown in the Figure below.



Figure 25: A shaft material with distributed load

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} \times (0.02)^4$$
$$= 7.855 \times 10^{-9} m^4$$

Since the density of the shaft material is $40 \times 10^3 kg/m^3$. Therefore mass of the shaft per length.

$$m_s = Area \times length \times density = \frac{\pi}{4}(0.02)^2 \times 1 \times 40 \times 10^3 = 12.6 kg/m$$

We know that static deflection due to 1 kg of mass at the center,

$$\delta = \frac{Wl^3}{48EI} = \frac{1 \times 9.81(0.6)^3}{48 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 28 \times 10^{-6} m$$

and static deflection due to mass of the shaft,

$$\delta_s = \frac{5Wl^4}{384EI} = \frac{5 \times 12.6 \times 9.81(0.6)^3}{384 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 0.133 \times 10^{-3} m$$

 \therefore Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta + \frac{\delta_s}{1.27}}} = \frac{0.4985}{\sqrt{28 \times 10^{-6} + \frac{0.133 \times 10^{-3}}{1.27}}} = \frac{0.4985}{11.52 \times 10^{-3}} = 43.3 Hz$$

Let N_c = Whirling speed of the shaft

We know that whirling speed of a shaft in r.p.s. is equal to the frequency of transverse vibration in Hz, therefore;

$$N_c = 43.3rps = 43.3 \times 60 = 2598rpm$$

Example 11: A shaft 1.5 m long supported in flexible bearings at the ends carries two wheels each of 50 kg mass. One wheel is situated at the center of the shaft and the other at a distance of 375 mm from the center towards left. The shaft is hollow of external diameter 75 mm and internal diameter 40 mm. The density of the shaft material is 7700 kg/m³ and its modulus of elasticity is 200 GN/m². Find the lowest whirling speed of the shaft, taking into account the mass of the shaft.

Solution: Given l = 1.5m; $m_1 = m_2 = 50kg$; $d_1 = 75mm = 0.075m$; $d_2 = 40mm = 0.04m$; $\rho = 7700 \ kg/m^3$; $E = 200 \ GN/m^2 = 200 \times 10^9 N/m^2$



Figure 26: A long supported shaft with flexible bearings

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} [(d_1)^4 - (d_2)^4] = \frac{\pi}{64} [(0.075)^4 - (0.04)^4] = 1.4 \times 10^{-6} m^4$$

Since the density of shaft material is 7700 kg/m³, therefore mass of the shaft per meter

length,
$$m_s = Arae \times length \times density$$

$$=\frac{\pi}{4}[(0.075)^2 - (0.04)^2] \times 1 \times 7700 = 24.34 kg/m$$

We know that the static deflection due to a load W is

$$=\frac{Wa^2b^2}{3EIl}=\frac{m.ga^2b^2}{3EIl}$$

 \therefore Static deflection due to a mass of 50kg at C,

$$\delta_1 = \frac{m_1 g a^2 b^2}{3EIL} = \frac{50 \times 9.81 (0.375)^2 (1.125)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 70 \times 10^{-6} m$$

...(Here
$$a = 0.375$$
 and $b = 1.124$)

Similarly, static deflection due to mass of 50kg at D

$$\delta_2 = \frac{m_1 g a^2 b^2}{3EIL} = \frac{50 \times 9.81 (0.75)^2 (0.75)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 123 \times 10^{-6} m$$

...(Here $a = b = 0.75$)

We know that static deflection due to uniformly distributed load or mass of the shaft

$$\delta_s = \frac{5}{384} \times \frac{wl^4}{El} = \frac{5}{384} \times \frac{24.34 \times 9.81(1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} m$$

...(Substituting, $w = m_s \times g$)

We know that frequency of transverse vibration

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \frac{\delta_s}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} \times 123 \times 10^{-6} + \frac{56 \times 10^{-6}}{1.27}}} Hz = 32.4 Hz$$

Since the whirling speed of the shaft (N_c) in r.p.s. is equal to the frequency of transverse vibration in Hz, thus $N_c = 32.4 r. p. s. = 32.4 \times 60 = 1944 r. p. m.$